2-1-1994

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Recommended Citation
Castro, Alfonso and Sudhasree Gadam. "Uniqueness of the stable and unstable positive solutions for semipositone problems", Nonlinear Analysis T.M.A., Vol. 22, No. 4 (1994), pp. 425-429.

Pergamon

Nonlinear Analysis, Theory, Methods & Applications, Vol. 22, No. 4, pp. 425–429, 1994

0362-546X/94 $6.00 + .00

UNIQUENESS OF STABLE AND UNSTABLE POSITIVE SOLUTIONS FOR SEMIPOSITONE PROBLEMS

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(*Received* 20 *October* 1992; *received for publication* 2 *June* 1993)

Key words and phrases: Semipositone problems, stable and unstable positive solutions.

1. INTRODUCTION

IN THIS PAPER we consider positive solutions to the equation

$$-\Delta u(x) = \lambda f(u(x)) \qquad \text{for } x \in \Omega \tag{1.1}$$

$$u(x) = 0 \qquad \text{for } x \in \partial\Omega \tag{1.2}$$

where Ω denotes the unit ball in $\mathbb{R}^N (N > 1)$, centered at the origin and $\lambda > 0$. Here f: $[0, \infty) \to \mathbb{R}$ is assumed to be monotonically increasing, concave and such that

$$f(0) < 0 \text{ (semipositone)}, \qquad f(u) > 0 \text{ for some } u > 0. \tag{1.3}$$

Let F be defined by $F(t) = \int_0^t f(s)\, ds$. We let β and θ denote the unique positive zeros of f and F, respectively. It can be easily seen that if u is a positive solution to (1.1), (1.2) then $u(0) > \theta$. We prove that for each λ there is at most one stable and one unstable positive solution to the problem (1.1), (1.2). More precisely, we prove the following theorem.

THEOREM A. If f is as above and $\lim_{t\to\infty} f'(t) = 0$, then there exist λ_1 and λ_2 with $\lambda_1 > \lambda_2$ such that for $\lambda > \lambda_1$ the problem (1.1), (1.2) has a unique positive solution, which is stable; for $\lambda \in (\lambda_2, \lambda_1]$ it has exactly two positive solutions, one stable and one unstable; and for $\lambda = \lambda_2$ it has a unique positive solution.

We note that the hypothesis $\lim_{t\to\infty} f'(t) = 0$ is also necessary for the existence of positive solutions for large values of λ (see [1, lemma 2.3]). The case $N = 1$ can be found in [2]. For other results in the radial case the reader is referred to [1, 3, 4]. In contrast with the positone case ($f(0) > 0$), theorem A shows the nonuniqueness of positive solutions for the semipositone case. In a forthcoming paper we prove that for any $\lambda > 0$, the problem (1.1), (1.2) has at most one positive solution when f is convex.

The positive solutions to (1.1), (1.2) are known to be radially symmetric (see [5]). Thus, the problem reduces to the study of solutions to the ordinary differential equation

$$u'' + ((N-1)/r)u' + \lambda f(u) = 0 \text{ in } (0, 1), \tag{1.4}$$

$$u'(0) = 0 \qquad \text{and} \qquad u(1) = 0, \tag{1.5}$$

where $'$ denotes differentiation with respect to $r = \|x\|$. Our proofs use the one-dimensional maximum principle, eigenvalue comparison arguments, and the bifurcation analysis at a degenerate solution.

2. FIRST AND SECOND VARIATIONS WITH RESPECT TO PARAMETERS

For any $d > 0$, let $u(r, \lambda, d)$ denote the solution to the initial value problem

$$u'' + ((N-1)/r)u' + \lambda f(u) = 0 \tag{2.1}$$

$$u'(0) = 0 \qquad \text{and} \qquad u(0) = d. \tag{2.2}$$

Thus, $u(r, \lambda, d)$ is a positive solution to (1.1), (1.2) if, in addition, it satisfies

$$u(r, \lambda, d) > 0 \qquad \text{for } r \in (0, 1) \text{ and } u(1, \lambda, d) = 0. \tag{2.3}$$

Let v denote the solution to the corresponding linearized problem

$$v'' + ((N-1)/r)v' + \lambda f'(u)v = 0 \tag{2.4}$$

$$v(0) = 1 \qquad \text{and} \qquad v'(0) = 0. \tag{2.5}$$

Let w denote the solution to the problem

$$w'' + ((N-1)/r)w' + \lambda f'(u)w = -\lambda f''(u)v^2 \tag{2.6}$$

$$w(0) = 0 \qquad \text{and} \qquad w'(0) = 0. \tag{2.7}$$

That is, v is the derivative of $u(r, \lambda, d)$ with respect to d and w is the second derivative of $u(r, \lambda, d)$ with respect to d.

LEMMA 1. If u is a solution to (2.1)–(2.3) then v has at most one zero in $[0, 1]$.

Proof. Let u be a solution to (2.1)–(2.3) such that v changes sign in $[0, 1]$. Let s be the first zero of v. That is $v > 0$ in $[0, s)$ and $v(s) = 0$. Let $r_0 \in (0, 1)$ be such that $u(r_0) = \beta$. Such an r_0 exists since $u(0) > \theta$ and $u(1) = 0$. We first prove that $s \notin (0, r_0)$. Suppose, on the contrary, that $s \in (0, r_0)$. By setting

$$\varphi(r) = v(r)/f(u(r)),$$

we obtain that in $(0, s)$, φ satisfies

$$\varphi'' + (((N-1)/r) + (2f'(u)u'/f(u)))\varphi' + (f''(u)(u')^2/f(u))\varphi = 0, \tag{2.8}$$

$$\varphi'(0) = 0, \qquad \varphi(0) > 0, \qquad \varphi(s) = 0. \tag{2.9}$$

Since $f(u(r)) > 0$ in $(0, s)$, and $f'' \leq 0$, by the maximum principle (see [6, theorem 4, p. 7]) we conclude that φ attains its maximum at 0 and $\varphi'(0) < 0$. Since this contradicts (2.9) we see that $s \geq r_0$.

Now we rule out the possibility of v having a second zero in $[r_0, 1]$. Suppose v has two zeros in $[r_0, 1]$. Let s_1 and s_2 be the first two zeros of v. Let $t \in (s_1, s_2)$ be such that $v'(t) = 0$ and $v' > 0$ in (t, s_2). Multiplying (2.4) by u' and integrating over (t, s_2) we get

$$u'(s_2)v'(s_2) - \int_t^{s_2} v'u'' + \int_t^{s_2} ((N-1)/r)v'u' + \lambda \int_t^{s_2} f'(u)u'v = 0.$$

This, with (2.1), yields

$$u'(s_2)v'(s_2) + 2\int_t^{s_2} ((N-1)/r)v'u' + \lambda \int_t^{s_2} (f(u)v)' = 0.$$

Hence, we get

$$u'(s_2)v'(s_2) - \lambda f(u(t))v(t) > 0,$$

which is a contradiction to our assumption that $t \in (r_0, 1)$. Thus v cannot have a second zero in $(0, 1]$. This proves the lemma. ■

LEMMA 2. If $u(\cdot, \lambda_0, d_0)$ is a solution to (2.1)–(2.3) and $v(1, \lambda_0, d_0) = 0$, then $u_\lambda(1, \lambda_0, d_0) < 0$ and $u_{dd}(1, \lambda_0, d_0) > 0$. Moreover, there exists an $\varepsilon > 0$ and a differentiable function $\Lambda: (d_0 - \varepsilon, d_0 + \varepsilon) \to \mathbb{R}$ such that for any $d \in (d_0 - \varepsilon, d_0 + \varepsilon)$, $u(\cdot, \Lambda(d), d)$ is a solution to (2.1)–(2.3), $\Lambda'(d_0) = 0$ and $\Lambda''(d_0) > 0$. In addition, if $u(1, \lambda, d) = 0$ with $|d - d_0| < \varepsilon$, $|\lambda - \lambda_0| < \varepsilon$ then $\lambda = \Lambda(d)$. In particular, if $|\lambda - \lambda^*| < \varepsilon$, $|d - d^*| < \varepsilon$ then $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$.

Proof. An elementary rescaling gives that

$$u(r/\rho, \lambda, d) = u(r, \lambda/\rho^2, d)$$

for all $\rho > 0$. Differentiating this with respect to ρ and evaluating at $\rho = 1$ we obtain

$$ru'(r, \lambda, d) = 2\lambda u_\lambda(r, \lambda, d). \tag{2.10}$$

By lemma 1, $v > 0$ on $[0, 1)$. Thus, v is an eigenfunction corresponding to the smallest eigenvalue of the problem

$$\begin{aligned} -\Delta\varphi - \lambda f'(u)\varphi &= \mu_i \varphi \qquad \text{in } \Omega, \\ \varphi &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

If $u'(1, \lambda, d_0) = 0$, then $\partial u/\partial x_i$, $1 \le i \le N$ are also eigenfunctions corresponding to the eigenvalue $\mu_1 = 0$. Since this contradicts the fact that μ_1 is simple we have $u'(1, \lambda_0, d_0) < 0$. This and (2.10) give $u_\lambda(1, \lambda_0, d_0) < 0$.

By the implicit function theorem there exists an $\varepsilon > 0$ and a differentiable function $\Lambda: (d_0 - \varepsilon, d_0 + \varepsilon) \to \mathbb{R}$ such that for any $d \in (d_0 - \varepsilon, d_0 + \varepsilon)$, $u(1, \Lambda(d), d) = 0$. In particular $u(\cdot, \Lambda(d), d)$ satisfies (2.1)–(2.3). Differentiating $u(1, \Lambda(d), d) = 0$ with respect to d, we have

$$u_\lambda(1, \Lambda(d_0), d_0)\Lambda'(d_0) + u_d(1, \Lambda(d_0), d_0) = 0. \tag{2.11}$$

This, (2.10) and the assumption $v(1, \lambda_0, d_0) = 0$ imply that $\Lambda'(d_0) = 0$. Differentiating again with respect to d, we obtain

$$u_\lambda(1, \Lambda(d_0), d_0)\Lambda''(d_0) + u_{dd}(1, \Lambda(d_0), d_0) = 0, \tag{2.12}$$

where we have used that $\Lambda'(d_0) = 0$. Multiplying (2.4) by $r^{N-1}w$ and (2.6) by $r^{N-1}v$, subtracting one from the other and integrating by parts on $[0, t)$ we see that

$$t^{n-1}(v'(t)w(t) - w'(t)v(t)) = \lambda \int_0^t r^{N-1} f''(u(r))v^3(r)\,\mathrm{d}r.$$

Now since by lemma 1 $u_d(r, \lambda_0, d_0) > 0$ on $[0, 1)$, it follows that $u_{dd}(1, \lambda_0, d_0) > 0$. This, in turn, implies that $\Lambda''(d_0) > 0$ which proves the lemma. ■

Lemma 3. If $\Gamma \subset \mathbb{R}^2$ is a component of $\{(\lambda, d): u(\cdot, \lambda, d)$ is a solution to (2.1)–(2.3)$\}$, then Γ is unbounded in the λ direction.

Proof. Let $(\lambda_0, d_0) \in \Gamma$. We distinguish three cases, namely $u_d(1, \lambda_0, d_0) > 0$, $u_d(1, \lambda_0, d_0) = 0$ and $u_d(1, \lambda_0, d_0) < 0$.

If $u_d(1, \lambda_0, d_0) > 0$, by the implicit function theorem there exists a $\delta > 0$ and an increasing differentiable function $\zeta: (\lambda_0 - \delta, \lambda_0 + \delta) \to \mathbb{R}$ such that

$$u(1, \lambda, \zeta(\lambda)) = 0. \tag{2.13}$$

Hence, $u(\cdot, \lambda, \zeta(\lambda))$ satisfies (2.1)–(2.3). We claim that ζ defined by (2.13) can be extended to (λ, ∞). In fact, suppose $\lambda^* = \sup\{\lambda: \zeta$ can be extended to (λ_0, λ) with $u(\cdot, s, \zeta(s))$ satisfying (2.1)–(2.3)$\} < \infty$. Letting $d^* := \sup\{\zeta(\lambda): \lambda < \lambda^*\}$, we see that $u_d(1, \lambda^*, d^*) = 0$. Taking ε as in lemma 2 we have a contradiction because if $|\lambda - \lambda^*| < \varepsilon$, $|d - d^*| < \varepsilon$ are such that $u(\cdot, \lambda, d)$ satisfies (2.1)–(2.3) then $\lambda > \lambda^*$. Thus, $\lambda^* = \infty$, which proves that Γ is unbounded in the λ direction.

If $u_d(1, \lambda_0, d_0) = 0$, applying lemma 2 we see that $u(\cdot, \Lambda(d_0 + \varepsilon/2), d_0 + \varepsilon/2)$ is in Γ and $u_d(1, \Lambda(d_0 + \varepsilon/2), d_0 + \varepsilon/2) > 0$. Thus, by the arguments in the latter paragraph, we conclude that Γ is unbounded in the λ direction.

Finally, if $u_d(1, \lambda_0, d_0) < 0$, then by the implicit function theorem there exists a $\delta > 0$ and a decreasing differentiable function $\eta: (\lambda_0 - \delta, \lambda_0 + \delta) \to \mathbb{R}$ such that $u(1, \lambda, \eta(\lambda)) = 0$ and $u(\cdot, \lambda, \eta(\lambda))$ satisfies (2.1)–(2.3). Let $\lambda^* = \inf\{\lambda: \eta$ can be extended to (λ, λ_0) with $u(\cdot, s, \eta(s))$ satisfying (2.1)–(2.3)$\}$. Letting $d^* = \sup\{\eta(\lambda): \lambda < \lambda_0\}$ we see that $u(\cdot, \lambda^*, d^*)$ satisfies (2.1)–(2.3). Because $f(0) < 0$ for $\lambda > 0$ near zero the problem (2.1)–(2.3) does not have a solution. In particular, $\lambda^* > 0$ and $d^* < \infty$. Since $u_d(1, \lambda^*, d^*) = 0$, arguing as in the previous case, we see that Γ is unbounded in the λ direction which proves the lemma. ■

3. PROOF OF THEOREM A

First we show that $\Gamma = \{(\lambda, d): u(\cdot, \lambda, d)$ satisfies (2.1)–(2.3)$\}$ is connected. In fact, if Γ_1 and Γ_2 are two connected components then by lemma 3 both contain elements of the form (λ, d) with $\lambda > 0$ large. However, by theorem A of [1] for λ large (2.1)–(2.3) has a unique solution. Since this contradicts that Γ_1 and Γ_2 are disjoint, we have $\Gamma_1 = \Gamma_2 = \Gamma$.

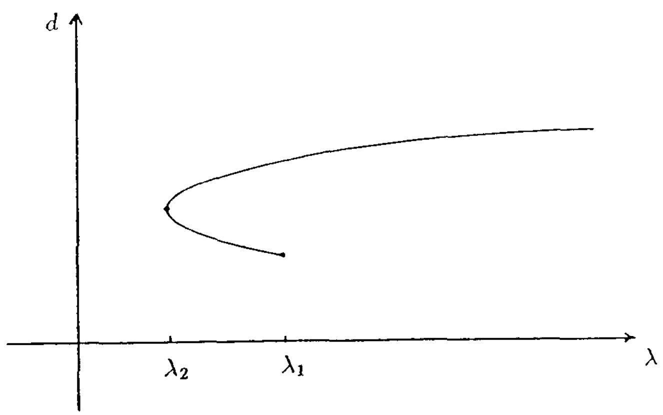

Fig. 1.

Next, we show that there exists a unique $(\lambda_2, d_2) \in \Gamma$ such that $u_d(1, \lambda, d) = 0$. Suppose, on the contrary, that there exist (λ_2, d_2) and (λ_2', d_2') on Γ with $v(1, \lambda_2, d_2) = v(1, \lambda_2', d_2') = 0$. Let $J = \gamma([0, 1])$ be an arc on Γ connecting (λ_2, d_2) and (λ_2', d_2') with $\gamma(0) = (\lambda_2, d_2)$ and $\gamma(1) = (\lambda_2', d_2')$. Let γ_1 and γ_2 denote the components of γ. Because J is compact, there are only finitely many points (λ, d) in J with $v(1, \lambda, d) = 0$. Let

$$t_1 = \min\{t \in (0, 1): v(1, \gamma_1(t), \gamma_2(t)) = 0\}$$

and let $\gamma(t_1) = (\lambda_2'', d_2'')$. Then for $t \in (0, t_1)$ either $v > 0$ or $v < 0$. Let us assume, for definiteness, that for $t \in (0, t_1)$ we have $v > 0$. On $\gamma((0, t_1))$, by the implicit function theorem, we have $d = \zeta(\lambda)$, with $\zeta' > 0$. Hence $\lambda_2 \neq \lambda_2''$. Suppose $\lambda_2 < \lambda_2''$. Then taking $\lambda_n \to \lambda_2''$ with $\lambda_n < \lambda_2''$ we see that $(\lambda_n, \zeta(\lambda_n)) \to (\lambda_2'', d_2'')$ which contradicts lemma 2. Similarly, $\lambda_2 > \lambda_2''$ also leads to a contradiction. Thus, we conclude that there exists a unique $(\lambda_2, d_2) \in \Gamma$ such that $u_d(1, \lambda, d) = 0$.

Now we prove that for each $\lambda > 0$ the problem (2.1)–(2.3) has at most one stable and one unstable solution. Suppose not. Let (λ_0, d_0) and (λ_0, d_0') be two points in Γ such that $v(1, \lambda_0, d_0) \cdot v(1, \lambda_0, d_0') > 0$. Let $K = \psi([0, 1])$ be a path in Γ connecting (λ_0, d_0) and (λ_0, d_0'); we also let ψ_1, ψ_2 denote the components of ψ. Without loss of generality we can assume that ψ is one to one. Because $v(1, \lambda_0, d_0) \cdot v(1, \lambda_0, d_0') > 0$, and ψ is one to one we see that there exists an $\varepsilon > 0$ such that $\psi_1((0, \varepsilon)) \subset (0, \lambda_0)$ or $\psi_1((0, \varepsilon)) \subset (\lambda_0, \infty)$. Let us assume that $\psi_1((0, \varepsilon)) \subset (0, \lambda_0)$; the other case can be treated in a similar way. Let t_1 be such that $\psi_1(t_1) = \min\{\psi(t): t \in (0, 1)\}$. Hence, $0 < \psi_1(t_1) < \lambda_0$. By the implicit function theorem $v(1, \psi_1(t_1), \psi_2(t_1)) = 0$. By lemma 2 there exist t_2 and t_3 near t_1 such that

$$v(1, \psi_1(t_2), \psi_2(t_2)) \cdot v(1, \psi_1(t_3), \psi_2(t_3)) < 0$$

and $t_2 < t_1 < t_3$. Without loss of generality we can assume that $v(1, \psi_1(t_2), \psi_2(t_2)) > 0$. By the intermediate value theorem there exists $t_4 \in (0, t_2)$ such that $v(1, \psi_1(t_4), \psi_2(t_4)) = 0$ which contradicts that Γ contains only one point (λ, d) with $v(1, \lambda, d) = 0$. This contradiction shows that for each λ the problem (2.1)–(2.3) has at most one stable and one unstable solution. ■

Acknowledgement—This work was partially supported by the Texas Advanced Research Program and NSF Grant DMS-8905936.

REFERENCES

1. Ali I., Castro A. & Shivaji R., Uniqueness and stability of nonnegative solutions for semipositone problems in a ball, *Proc. Am. math. Soc.* **117**(3), 775–782 (1993).
2. Castro A. & Shivaji R., Nonnegative solutions for a class of nonpositone problems, *Proc. R. Soc. Edinb.* **108A**, 291–302 (1988).
3. Castro A., Garner J. & Shivaji R., Existence results for classes of sublinear semipositone problems, *Result. Math.* **23**, 214–220 (1993).
4. Castro A. & Shivaji R., Nonnegative solutions for a class of radially symmetric nonpositone problems, *Proc. Am. math. Soc.* **106**(3), 735–740 (1989).
5. Gidas B., Ni W. & Nirenberg L., Symmetry and related properties via the maximum principles, *Communs math. Phys.* **68**, 209–243 (1979).
6. Protter M. H. & Weinberger H. F., *Maximum Principles in Differential Equations*. Prentice-Hall, Englewood Cliffs, New Jersey (1967).